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STABILITY OF BIORTHOGONAL WAVELET BASES

PAUL F. CURRAN

Department of Electronic and Electrical Engineering,
University College Dublin, Belfield, Dublin 4, Ireland
paul.curran@ucd.ie

GARY MCDARBY

Medialab Europe,
Crane Street, Dublin 8, Ireland
gary@media.mit.edu

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We investigate the lifting scheme as a method for constructing compactly supported biorthogonal scaling functions and wavelets. A well-known issue arising with the use of this scheme is that the resulting functions are only formally biorthogonal. It is not guaranteed that the new wavelet bases actually exist in an acceptable sense. To verify that these bases do exist one must test an associated linear operator to ensure that it has a simple eigenvalue at one and that all its remaining eigenvalues have modulus less than one, a task which becomes numerically intensive if undertaken repeatedly. We simplify this verification procedure in two ways: (i) we show that one need only test an identifiable half of the eigenvalues of the operator, (ii) we show that when the operator depends upon a single parameter, the test first fails for values of that parameter at which the eigenvalue at one becomes a multiple eigenvalue. We propose that this new verification procedure comprises a first step towards determining simple conditions, supplementary to the lifting scheme, to ensure existence of the new wavelets generated by the scheme and develop an algorithm to this effect.

Keywords: Lifting scheme; biorthogonal wavelets; stability.

AMS Subject Classification: 15A18, 15A48, 42C40

1. Introduction

In 1988 Daubechies[3] discovered a class of compactly supported orthonormal wavelet bases for $L_2(R)$, that is to say orthonormal bases comprising integer shifts and dyadic scales of a single, fixed function, this function being identically zero outside a certain closed, bounded interval. Prior to this discovery the only known compactly supported wavelet basis for $L_2(R)$ was that due to Haar[3] which it transpires is the first member of the Daubechies class of wavelet bases. Given a particular signal in $L_2(R)$ and a particular wavelet basis for $L_2(R)$ one naturally seeks to *analyse* the signal, i.e. to determine the coefficients of the expansion of the signal relative to the basis. Equally, given the coefficients one seeks to *synthesise* the signal, i.e. to reconstruct the signal as a weighted sum of basis elements. Towards this end Mallat[6] identified the relationship between wavelet transforms and multiresolution analyses and showed that analysis and synthesis relative to an orthonormal wavelet basis can be achieved using orthogonal filter banks. Researchers in the field of signal processing had been employing orthogonal filter banks for a number of years.[11,12] The merging of these approaches led to a rich cross-fertilisation of ideas and to practical numerical algorithms for the evaluation of discrete wavelet transforms and hence for the application of wavelets to signal processing.

While it is well known that orthogonality of the basis is a useful property in the analysis and synthesis of signals, this property is dispensable. In 1992 Cohen, Daubechies and Feauveau[2] introduced the idea of biorthogonal wavelet bases. In this case two distinct bases are employed, one for analysis (i.e. for the determination of wavelet coefficients) and one for synthesis (i.e. for the reconstruction of signal). The two bases are not necessarily orthogonal in their own right but are orthogonal to one another. Cohen *et al.*[2] also demonstrated that filter banks may again be employed to implement a discrete wavelet transform in this more general setting but that the filter banks are no longer orthogonal, they are biorthogonal. Biorthogonal bases offer increased flexibility in the design of the associated filter bank enabling, for example, the construction of filter banks from linear phase filters. This is possible in the case of orthogonal bases only when the Haar filter bank is employed.[14] Cohen, Daubechies and Feauveau,[2] Cohen and Daubechies[1] and Strang[7] provide necessary and sufficient conditions for a pair of dual filters to generate biorthogonal compactly supported scaling functions and wavelets in $L_2(R)$.

From 1996 Sweldens,[8–10] in light of the work of Vetterli and Herley,[13] introduced the lifting scheme as a versatile method for the construction of biorthogonal scaling functions and wavelets. Given a starting set of scaling functions and wavelets the scheme generates a new set of scaling functions and wavelets. A well-known issue with the scheme is that, whereas it ensures that the new wavelets are formally biorthogonal, it does not guarantee that they exist in a satisfactory sense (namely that they form a Riesz basis in $L_2(R)$). Accordingly, if one employs these new wavelets one may be led to the situation in which one is employing filter banks to analyse and synthesise signals relative to bases which may not exist. As a consequence, while one will in principle always achieve perfect reconstruction with biorthogonal filter banks (i.e. analysis followed by synthesis perfectly reproduces the input signal), in practice the process of analysis followed by synthesis becomes unstable relative to the filter coefficients (i.e. small errors in filter coefficients or round-off errors in internal calculations result in a loss of the perfect reconstruction property). To verify that the new wavelets do exist in $L_2(R)$ and that they form a Riesz basis, one must test the conditions of Cohen, Daubechies and Feauveau. In one form[1] this involves testing a linear operator (the transition operator) associated with a new dual filter generated by the scheme to ensure that it has a simple eigenvalue at one and that all its remaining eigenvalues have modulus less than one. We refer to this condition on the eigenvalues of this operator as condition E. We restrict our attention to the case where all filters are finite (FIR filters) as this is arguably the only important case in applications. In this case the transition operator is represented by a finite square matrix which we call the generalised Lawton matrix. The task of testing condition E becomes numerically intensive if undertaken repeatedly. Such repetitive testing occurs for example in the case where the user of the lifting scheme does not know *a priori* which lifting filter will yield the best new wavelets and accordingly elects to try many different lifting filters and subsequently compare the performance of the numerous new filters.

The purpose of this paper is to develop a simpler, but equivalent, set of conditions which permit more rapid numerical testing. Towards this end we make two observations concerning condition E in the case of finite filters satisfying some minor restrictions which we elucidate below: (i) Condition E holds for a generalised Lawton matrix if and only if it holds for a generally constructible matrix of about half the size. Below we call the eigenvalues of this reduced order matrix the symmetric eigenvalues of the generalised Lawton matrix. As a result of this observation the numerical effort involved in testing condition E can be essentially halved. (ii) If the generalised Lawton matrix depends continuously on some parameters then condition E first fails for values of these parameters for which the reduced order matrix of observation (i) has an eigenvalue at one of multiplicity greater than one. One would expect from the nature of condition E that it first fails when the generalised Lawton matrix acquires a second eigenvalue of modulus one. Not so, according to observation (ii). The mechanism by which condition E can first fail is far more specific, it first fails when the matrix acquires a second eigenvalue *equal to one* and moreover, this eigenvalue is symmetric.

Whereas the lifting scheme in general contains many free parameters we essentially reformulate it in terms of a single parameter in order to exploit the previous observations concerning condition E. The principle contribution of the present work is the observation that for real, finite filters this single-parameter reformulation of the lifting scheme generates biorthogonal filter banks having associated wavelets in $L_2(R)$ provided the single parameter lies in an open interval containing zero. Moreover, we provide an algorithm for finding this interval. As a supplement to the lifting scheme this algorithm provides a method for constructing, with a single test, a *whole class* of biorthogonal filter banks with associated wavelets in $L_2(R)$. It is in this capacity to generate *parametrised classes* of well-behaved filter banks that the proposed scheme most clearly reveals a superiority of numerical performance over existing schemes.

2. Generalised Lawton Matrix

Cohen and Daubechies[1] and Cohen, Daubechies and Feaveau[2] establish necessary and sufficient conditions for a pair of dual real, finite filters to generate biorthogonal

Riesz bases of compactly supported wavelets in $L_2(R)$. In accordance with this work, given real finite sequences $h = (h_n)$ and $\tilde{h} = (\tilde{h}_n)$, having supports contained in $\{-M, \ldots, M\}$ and $\{-\tilde{M}, \ldots, \tilde{M}\}$ respectively, we define 2π-periodic functions

$$\hat{h}(\zeta) = \sum_{n=-M}^{M} h_n e^{-in\zeta}\,, \quad \hat{\tilde{h}}(\zeta) = \sum_{n=-\tilde{M}}^{\tilde{M}} \tilde{h}_n e^{-in\zeta}\,. \tag{2.1}$$

We assume that $\hat{h}(0) = \hat{\tilde{h}}(0) = \sqrt{2}$ and that $\hat{h}(\pi) = \hat{\tilde{h}}(\pi) = 0$. We also impose the constraint

$$\hat{\tilde{h}}(\zeta)\overline{\hat{h}(\zeta)} + \hat{\tilde{h}}(\zeta+\pi)\overline{\hat{h}(\zeta+\pi)} = 2 \tag{2.2}$$

as a necessary condition for biorthogonality.[8] We define two associated transition operators[1,2] acting on the 2π-periodic functions $\hat{f}$ as

$$\begin{aligned}(P_0 f)(\zeta) &= \frac{1}{2}\left|\hat{h}\left(\frac{\zeta}{2}\right)\right|^2 \hat{f}\left(\frac{\zeta}{2}\right) + \frac{1}{2}\left|\hat{h}\left(\frac{\zeta}{2}+\pi\right)\right|^2 \hat{f}\left(\frac{\zeta}{2}+\pi\right)\,,\\ (\tilde{P}_0 f)(\zeta) &= \frac{1}{2}\left|\hat{\tilde{h}}\left(\frac{\zeta}{2}\right)\right|^2 \hat{f}\left(\frac{\zeta}{2}\right) + \frac{1}{2}\left|\hat{\tilde{h}}\left(\frac{\zeta}{2}+\pi\right)\right|^2 \hat{f}\left(\frac{\zeta}{2}+\pi\right)\,.\end{aligned} \tag{2.3}$$

Let T_{2M} denote the set of trigonometric polynomials of period 2π having the form $\sum_{n=-2M}^{2M} f_n e^{-in\zeta}$. As noted by Cohen and Daubechies[1] T_{2M} is invariant under P_0 and the restriction of operator P_0 to T_{2M} can be represented (relative to the standard Fourier basis) by the $(4M+1)\times(4M+1)$ matrix Λ where

$$[\Lambda]_{kl} = \sum_q h_q h_{q+l-2k} k, l \in \{-2M, \ldots, 2M\} \tag{2.4}$$

and where the sequence h is extended to an infinite sequence by appending zeros. In line with the comments of Cohen and Daubechies,[2] we will refer to matrix Λ as the *generalised Lawton matrix* associated with the sequence h. Letting $\eta_k = \sum_q h_q h_{q+k}$ for $k \in \{-2M, \ldots, 2M\}$ and assuming that sequence h is real, the associated generalised Lawton matrix takes the form

$$\Lambda = \begin{pmatrix} \eta_{2M} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \eta_{2M-2} & \eta_{2M-1} & \eta_{2M} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \eta_{2M} & \eta_{2M-1} & \eta_{2M-2} & \cdots & \eta_1 & \eta_0 & \eta_1 & \cdots & \eta_{2M-2} & \eta_{2M-1} & \eta_{2M} \\ 0 & 0 & \eta_{2M} & \cdots & \eta_3 & \eta_2 & \eta_1 & \cdots & \eta_{2M-4} & \eta_{2M-3} & \eta_{2M-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \eta_{2M} \end{pmatrix}. \tag{2.5}$$

As discussed by Cohen and Daubechies[1] it follows from the assumption, $\hat{h}(0) = \sqrt{2}$ and $\hat{h}(\pi) = 0$, that the elements in each column of Λ sum to one. We will refer to this condition as the *column sum condition*. Similar comments apply to the transition operator $\tilde{P}_0$. The following result is well known[1,7]:

Theorem 2.1. *A pair of real, finite sequences, $(h, \tilde{h})$, satisfying $\hat{h}(0) = \hat{\tilde{h}}(0) = \sqrt{2}$, $\hat{h}(\pi) = \hat{\tilde{h}}(\pi) = 0$ and $\hat{\tilde{h}}(\zeta)\overline{\hat{h}(\zeta)} + \hat{\tilde{h}}(\zeta+\pi)\overline{\hat{h}(\zeta+\pi)} = 2$ generate biorthogonal Riesz bases of compactly supported wavelets if and only if the generalised Lawton matrix associated with each sequence has one eigenvalue at* 1 *of algebraic multiplicity one and all remaining eigenvalues have modulus less than one.*

We refer to the eigenvalue constraint imposed upon the generalised Lawton matrices associated with the sequences in Theorem 2.1 as condition E.

3. Eigenvalue Classification

We present some well-known properties of the transition operators associated with real, finite filters in the form of an explicit decomposition of the corresponding generalised Lawton matrix.

Let E denote the set of even trigonometric polynomials of period 2π. Clearly E is invariant under transition operator P_0. Let O denote the set of odd trigonometric polynomials of period 2π. Again it is clear that O is invariant under P_0. It follows that the sets $E \cap T_{2M}$ and $O \cap T_{2M}$ are invariant under P_0. We may readily conclude that the eigenvalues of the restriction of P_0 to T_{2M} (i.e. the eigenvalues of the generalised Lawton matrix) fall into two classes: (i) the eigenvalues of the restriction of P_0 to $E \cap T_{2M}$ (of which there are clearly $2M+1$) and (ii) the eigenvalues of the restriction of P_0 to $O \cap T_{2M}$ (of which there are clearly $2M$). We will refer to the eigenvalues of the first class as the *symmetric eigenvalues* of Λ and to the eigenvalues of the second class as the *skew-symmetric eigenvalues* of Λ. To extract the symmetric and skew-symmetric eigenvalues of the generalised Lawton matrix we shall require an explicit decomposition of the matrix. Towards that end we define

$$J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in R^{n \times n} . \tag{3.6}$$

A vector $v \in C^{4M+1}$ will be said to be symmetric if $J_{4M+1}v = v$ or skew-symmetric if $J_{4M+1}v = -v$. It is not difficult to see that an eigenvalue of a generalised Lawton matrix Λ is symmetric if and only if it has an associated symmetric eigenvector. Similarly an eigenvalue of Λ is skew-symmetric if and only if it has an associated skew-symmetric eigenvector. We observe the following result whose proof, with reference to (2.5), is trivial:

Lemma 3.1. *The generalised Lawton matrix associated with a real, finite sequence h whose support is contained in $\{-M, \ldots, M\}$ has the following structure*:

$$\Lambda = \begin{pmatrix} A & a & B \\ b^{\mathrm{T}} & c & b^{\mathrm{T}} J_{2M} \\ J_{2M} B J_{2M} & J_{2M} a & J_{2M} A J_{2M} \end{pmatrix}, \tag{3.7}$$

where $A, B \in R^{2M\times 2M}$, $a, b \in R^{2M}$ and $c \in R$.

Employing this result we obtain the following:

Lemma 3.2. *The eigenvalues of the generalised Lawton matrix Λ associated with a real, finite sequence h whose support is contained in $\{-M, \ldots, M\}$ and which satisfies $\hat{h}(0) = \sqrt{2}$ and $\hat{h}(\pi) = 0$ may be classified as follows*:

(i) *One eigenvalue is symmetric, equals 1 and has an associated left eigenvector $[1, \ldots, 1]$.*
(ii) *A further $2M$ eigenvalues are symmetric and are the eigenvalues of the reduced order matrix $(A + BJ_{2M} - 2a[1, \ldots, 1])$.*
(iii) *The remaining $2M-1$ eigenvalues are skew-symmetric and are the eigenvalues of the reduced order matrix $(A - BJ_{2M})$ with one of them being $1/2$ and having associated left eigenvector $[2M, \ldots 1, 0, -1, \ldots, -2M]$.*

Proof. As $\hat{h}(0) = \sqrt{2}$ and $\hat{h}(\pi) = 0$ we know that Λ satisfies the column sum condition. Under these conditions Strang[7] shows that row vectors $[1, \ldots, 1]$ and $[2M, \ldots 1, 0, -1, \ldots, -2M]$ are left eigenvectors of Λ with associated eigenvalues 1 and $1/2$ respectively. We note that any vector may be uniquely expressed as the sum of a symmetric vector and a skew-symmetric vector. On considering the problem of determining for a given vector these symmetric and skew-symmetric *components* we are quickly led to the following matrix of change of basis

$$H = \begin{pmatrix} \frac{1}{2} I_{2M} & 0 & \frac{1}{2} J_{2M} \\ 0^{\mathrm{T}} & 1 & 0^{\mathrm{T}} \\ \frac{1}{2} I_{2M} & 0 & -\frac{1}{2} J_{2M} \end{pmatrix}, \tag{3.8}$$

where I_{2M} is the $2M \times 2M$ identity matrix. From (3.7) it readily follows that

$$H\Lambda H^{-1} = \begin{pmatrix} (A + BJ_{2M}) & a & 0 \\ 2b^{\mathrm{T}} & c & 0^{\mathrm{T}} \\ 0 & 0 & (A - BJ_{2M}) \end{pmatrix}. \tag{3.9}$$

This explicit decomposition confirms that the eigenvalues of Λ fall into two categories. The first category comprises the eigenvalues of the lower $2M \times 2M$ block $(A - BJ_{2M})$ of $H\Lambda H^{-1}$. It is elementary to show that, when the column sum condition holds, $[2M, (2M-1), \ldots, 1]$ is a left eigenvector of $(A - BJ_{2M})$ with associated eigenvalue 1/2. Given the nature of the matrix of change of basis it is clear that these are the skew-symmetric eigenvalues of Λ. The second category comprises the eigenvalues of the upper $(2M+1)\times(2M+1)$ block of $H\Lambda H^{-1}$. Given the nature of the matrix of change of basis it can readily be shown that these are the symmetric eigenvalues of Λ. To provide a further refinement of this decomposition consider the $(2M+1)\times(2M+1)$ matrix

$$Q = \begin{pmatrix} I_{2M} & 0 \\ 1,\dots,1 & 1 \end{pmatrix} . \tag{3.10}$$

By employing the column sum condition, it is elementary to show that

$$Q\begin{pmatrix} (A+BJ_{2M}) & a \\ 2b^{\mathrm{T}} & c \end{pmatrix} Q^{-1} = \begin{pmatrix} (A+BJ_{2M}-2a[1,\dots,1]) & 2a \\ 0^{\mathrm{T}} & 1 \end{pmatrix} . \tag{3.11}$$

Since the eigenvalues of this matrix are the symmetric eigenvalues of Λ it follows that these eigenvalues fall into two sub-categories when the column sum condition holds. The first sub-category comprises the single eigenvalue at 1 whose existence is guaranteed by the column sum condition.[7] The second sub-category comprises all $2M$ remaining symmetric eigenvalues, these being the eigenvalues of the matrix $(A+BJ_{2M}-2a[1,\dots,1])$. □

We call the guaranteed eigenvalue at 1 the *symmetric eigenvalue of type* (1) and that at 1/2 the *skew-symmetric eigenvalue of type* (1). The remaining $2M$ symmetric eigenvalues we call the *symmetric eigenvalues of type* (2) and the remaining $(2M-1)$ skew-symmetric eigenvalue we call the *skew-symmetric eigenvalues of type* (2).

4. Non-Negative Matrices

Let S_{2M} denote the set of trigonometric polynomials in T_{2M} having real coefficients. The set S_{2M} is invariant under P_0. Any 2π-periodic, real trigonometric polynomial $\hat{f}$ is non-negative, denoted $\hat{f}\geq 0$, if:

$$\mathrm{Re}(\hat{f}(\zeta)) + \mathrm{Im}(\hat{f}(\zeta)) \geq 0 \quad \text{for all } \zeta\in[0,2\pi] . \tag{4.12}$$

If F_+ denotes the set of all non-negative, 2π-periodic, real trigonometric polynomials, then it is apparent that F_+ is invariant under the transition operator P_0. Accordingly $F_+\cap S_{2M}$ is also invariant under P_0. In other words, the restriction of P_0 to S_{2M} is a non-negative linear operator. It is clear that relative to the Fourier basis this restriction is represented by the generalised Lawton matrix Λ.

Krein and Rutman[5] define a convex cone in a finite dimensional, real vector space to be a subset, K, of the vector space having the following properties:

(i) If $x\in K$, then $\alpha x\in K$ for all scalars $\alpha\geq 0$.
(ii) If $x,y\in K$, then $x+y\in K$.

(iii) If $x, y \in K$, $x \neq 0$, then $x + y \neq 0$.
(iv) K is closed relative to the standard Euclidean norm-topology on the vector space.

Lemma 4.1. *$F_+ \cap S_{2M}$ is a convex cone in the sense of Krein and Rutman.*

Proof. It is trivial to show that $F_+ \cap S_{2M}$ possesses properties (i), (ii) and (iv) of a convex cone. As regards property (iii) assume that both x and $-x$ are in $F_+ \cap S_{2M}$ for some x. One may conclude that:

$$\mathrm{Re}(x(\zeta)) + \mathrm{Im}(x(\zeta)) = 0 \quad \text{for all } \zeta \in [0, 2\pi]\,. \tag{4.13}$$

It follows by elementary Fourier theory that $x = 0$ and property (iii) is a consequence of this observation. □

Lemma 4.2. $(F_+ \cap S_{2M}) + (-(F_+ \cap S_{2M})) = S_{2M}$.

Proof. Since $F_+ \cap S_{2M}$ is a convex cone $(F_+ \cap S_{2M}) + (-(F_+ \cap S_{2M}))$ is a subspace of S_{2M}. If it does not equal S_{2M}, then there must exist a nonzero $y \in S_{2M}$ orthogonal to the subspace $(F_+ \cap S_{2M}) + (-(F_+ \cap S_{2M}))$, i.e. $\int_0^{2\pi} \overline{y(\zeta)} \nu(\zeta) d\zeta = 0$ for all $\nu \in F_+ \cap S_{2M}$. By selecting in turn $\nu(\zeta) = 1, 1 + \cos(\zeta), 1 + i\sin(\zeta), 1 + \cos(2\zeta), 1 + i\sin(2\zeta), \ldots$ one concludes that the coefficients of y are all zero. Hence $y = 0$ in contradiction of its definition. The result follows from this contradiction. □

The previous results permit a number of corollaries. Points (i), (ii) and (iii) of the corollaries follow without difficulty from Lemma 4.1 and the properties of P_0. Point (iv) requires justification which we elect to provide for Corollary 4.2 only. Let $L_{2M} = E \cap S_{2M}$, i.e. L_{2M} is the set of all real, even, 2π-periodic trigonometric polynomials with maximum frequency $2M$.

Corollary 4.1. (i) *L_{2M} is a subspace of S_{2M}.* (ii) *$F_+ \cap L_{2M}$ is invariant under P_0.* (iii) *$F_+ \cap L_{2M}$ is a convex cone in L_{2M}.* (iv) *$(F_+ \cap L_{2M}) + (-(F_+ \cap L_{2M})) = L_{2M}$.*

Let $Z_0 = \{\hat{f} \in S_{2M} : \hat{f}(0) = 0, \hat{f}'(0) = 0\}$ where $\hat{f}'$ denotes the derivative of $\hat{f}$.

Corollary 4.2. (i) *Z_0 is a subspace of S_{2M}.* (ii) *$F_+ \cap Z_0$ is invariant under P_0.* (iii) *$F_+ \cap Z_0$ is a convex cone in Z_0.* (iv) *$(F_+ \cap Z_0) + (-(F_+ \cap Z_0)) = Z_0$.*

Proof. It is clear that $(F_+ \cap Z_0) + (-(F_+ \cap Z_0))$ is a subspace of Z_0. Consequently, if assertion (iv) does not hold, then there exists a nonzero $y \in Z_0$ orthogonal to this subspace, i.e. $\int_0^{2\pi} \overline{y(\zeta)} \nu(\zeta) d\zeta = 0$ for all $\nu \in F_+ \cap Z_0$. Let $y(\zeta) = \sum_{=-2M}^{2M} y_n e^{in\zeta}$, y_n real for each n. By making in turn the selections $\nu(\zeta) = 1 - \cos(\zeta), 1 - \cos(2\zeta), \ldots \in F_+ \cap Z_0$ one obtains:

$$y_0 = \frac{y_{-1} + y_1}{2} = \frac{y_{-2} + y_2}{2} = \cdots\,. \tag{4.14}$$

As $y \in Z_0$, it follows by definition that:

$$y(0) = 0 = y_0 + y_{-1} + y_1 + y_{-2} + y_2 + \cdots . \tag{4.15}$$

Hence,

$$y_0 = 0\,, \quad y_{-1} = -y_1\,, \quad y_{-2} = y_2, \ldots . \tag{4.16}$$

By selecting $\nu(\zeta) = \sqrt{2}((\sqrt{2} - e^{ik\zeta})(1 - \cos(\zeta))) \in F_+ \cap Z_0$ for each integer $k \geq 1$, it follows from (4.16) that $y_{k+1} = 2y_k - y_{k-1}$. Hence $y_k = ky_1$ for each $k \geq 1$. But $y \in Z_0$ implies that $y'(0) = 0$ from which it now follows that $y_1 = 0$ and in consequence that $y = 0$. □

Corollary 4.3. (i) *$Z_0 \cap L_{2M}$ is a subspace of S_{2M}.* (ii) *$F_+ \cap Z_0 \cap L_{2M}$ is invariant under P_0.* (iii) *$F_+ \cap Z_0 \cap L_{2M}$ is a convex cone in $Z_0 \cap L_{2M}$.* (iv) *$(F_+ \cap Z_0 \cap L_{2M}) + (-(F_+ \cap Z_0) \cap L_{2M}) = Z_0 \cap L_{2M}$.*

Finally we note the following readily verified result:

Lemma 4.3. *There exists no nonzero, odd, non-negative, real trigonometric polynomial in S_{2M}.*

5. Generalised Frobenius–Perron Theory

In their celebrated treatise, Krein and Rutman[5] present a generalisation of the classical Frobenius–Perron theorem which we may paraphrase as follows:

Theorem 5.1. *Let K be a convex cone with non-null interior in a real, finite-dimensional vector space; if a linear mapping M maps K into itself and is not nilpotent, then there is a real, positive eigenvalue λ_K of M with an associated eigenvector lying in K, having the property that no other eigenvalue of M has modulus exceeding λ_K.*

By employing the results of Sec. 4 together with Theorem 5.1, we may make a number of assertions concerning generalised Lawton matrices associated with real, finite sequences.

Lemma 5.1. *Let Λ be the generalised Lawton matrix associated with a real, finite sequence h which satisfies $\hat{h}(0) = \sqrt{2}$, $\hat{h}(\pi) = 0$, then there exists a real, positive eigenvalue, L, of Λ such that:* (i) *all remaining eigenvalues of Λ have modulus less than or equal to L,* (ii) *there exists a real, non-negative eigenfunction, v_L, of P_0 associated with eigenvalue L.*

Proof. We have noted previously that P_0 maps $F_+ \cap S_{2M}$ into itself. By Lemmas 4.1 and 4.2, $F_+ \cap S_{2M}$ is a convex cone with non-null interior. P_0 is not nilpotent as it has an eigenvalue equal to 1. Result follows from Theorem 5.1 and definition of Λ. □

Lemma 5.2. *Let* Λ *be the generalised Lawton matrix associated with a real, finite sequence* h *which satisfies* $\hat{h}(0) = \sqrt{2}$, $\hat{h}(\pi) = 0$, *then there exists a real, positive symmetric eigenvalue*, S, *of* Λ *such that*: (i) *all remaining symmetric eigenvalues of* Λ *have modulus less than or equal to* S, (ii) *there exists a real, non-negative eigenfunction*, v_S, *of* P_0 *associated with eigenvalue* S.

Proof. By Corollary 4.1, P_0 maps $F_+ \cap L_{2M}$ into itself and $F_+ \cap L_{2M}$ is a convex cone with non-null interior. The restriction of P_0 to subspace L_{2M} is not nilpotent as P_0 has a symmetric eigenvalue equal to 1. Result follows from Theorem 5.1 and the fact that the eigenvalues of the restriction of P_0 to subspace L_{2M} are the symmetric eigenvalues of Λ. □

Lemma 5.3. *Let* Λ *be the generalised Lawton matrix associated with a real, finite sequence* h *which satisfies* $\hat{h}(0) = \sqrt{2}$, $\hat{h}(\pi) = 0$, *then there exists a real, positive eigenvalue*, ρ, *of* Λ *which is either symmetric of type* (2) *or skew-symmetric of type* (2) *such that* (i) *all remaining symmetric and skew-symmetric eigenvalues of type* (2) *of* Λ *have modulus less than or equal to* ρ, (ii) *there exists a real, non-negative eigenfunction*, v_ρ, *of* P_0 *associated with eigenvalue* ρ.

Proof. By Corollary 4.2 P_0 maps $F_+ \cap Z_0$ into itself and $F_+ \cap Z_0$ is a convex cone with non-null interior. If the restriction of P_0 to subspace Z_0 is nilpotent then, by definition, all of the symmetric and skew-symmetric eigenvalues of type (2) equal zero. It follows that Λ has one eigenvalue at 1, one eigenvalue at 1/2 and $(4M-1)$ eigenvalues at 0. Letting $\eta_k = \sum_q h_q h_{q+k}$, it is trivial to show that Λ has the form of (2.5) and therefore has a double eigenvalue at η_{2M}. Accordingly η_{2M} must be zero. In consequence, Λ has a double eigenvalue at η_{2M-1} and, therefore η_{2M-1} must also be zero. Proceeding in this manner one concludes that $\eta_{2M} = \eta_{2M-1} = \cdots = \eta_1 = 0$. It is elementary to show that Λ cannot satisfy the column sum condition in this case. Hence the restriction of P_0 to subspace Z_0 is not nilpotent. Result follows from Theorem 5.1 and the readily proven fact that the eigenvalues of the restriction of P_0 to subspace Z_0 are the symmetric eigenvalues of type (2) and the skew-symmetric eigenvalues of type (2) of Λ. □

Lemma 5.4. *Let* Λ *be the generalised Lawton matrix associated with a real, finite sequence* h *which satisfies* $\hat{h}(0) = \sqrt{2}$, $\hat{h}(\pi) = 0$, *then there exists a real, positive eigenvalue*, σ, *of* Λ *which is symmetric of type* (2) *such that*: (i) *all remaining symmetric eigenvalues of type* (2) *of* Λ *have modulus less than or equal to* σ, (ii) *there exists a non-negative eigenfunction*, v_σ, *of* P_0 *associated with eigenvalue* σ.

Proof. By Corollary 4.3 P_0 maps $F_+ \cap Z_0 \cap L_{2M}$ into itself and $F_+ \cap Z_0 \cap L_{2M}$ is a convex cone with non-null interior. If the restriction of P_0 to subspace $Z_0 \cap L_{2M}$ is nilpotent then, by definition, all of the symmetric eigenvalues of type (2) equal zero. It follows that the eigenvalue ρ of Lemma 5.3 must be skew-symmetric. Lemma 5.3

also assures the existence of an associated real, non-negative eigenfunction v_ρ of P_0 that must therefore be odd and nonzero. Lemma 4.3 denies the existence of such an eigenfunction. The restriction of P_0 to subspace $Z_0 \cap L_{2M}$ must therefore not be nilpotent. Result follows from Theorem 5.1 and the readily proven fact that the eigenvalues of the restriction of P_0 to subspace $Z_0 \cap L_{2M}$ are the symmetric eigenvalues of type (2) of Λ. □

Lemma 4.3 as it is employed in the proof of Lemma 5.4 permits us to make a number of observations concerning the eigenvalues L, S, ρ and σ of Lemmas 5.1–5.4.

Lemma 5.5. *Eigenvalue L is symmetric and equals eigenvalue S. Eigenvalue ρ is symmetric of type* (2) *and equals eigenvalue σ.*

Proof. Lemma 5.1 assures that eigenfunction v_L is real, nonzero and non-negative. By Lemma 4.3 this function cannot, therefore, be odd. Hence eigenvalue L cannot be skew-symmetric and must, therefore, be symmetric. It is now trivial to show that $L = S$.

Lemma 5.3 assures that eigenfunction v_ρ is real, nonzero and non-negative. As above, Lemma 4.3 asserts that this function cannot be odd and, therefore, that eigenvalue ρ cannot be skew-symmetric. Hence ρ must be symmetric of type (2) and it is now trivial to show that $\rho = \sigma$. □

We note that the eigenvalue σ, whose existence is guaranteed by Lemma 5.4, is uniquely defined by the generalised Lawton matrix (and hence by the sequence associated with it). We are finally in a position to state and prove the primary result of this investigation, which amounts to a reformulation of the eigenvalue condition, condition E, of Theorem 2.1.

Theorem 5.2. *The generalised Lawton matrix associated with a real, finite sequence h for which $\hat{h}(0) = \sqrt{2}$, $\hat{h}(\pi) = 0$, satisfies condition E if and only if the particular eigenvalue σ is less than one.*

Proof. The conditions imposed imply that the generalised Lawton matrix associated with sequence h is real and satisfies the column sum condition. Hence, the division of eigenvalues into symmetric eigenvalues of type (1) and (2) and skew-symmetric eigenvalues of type (1) and (2) is valid. If the particular eigenvalue σ is greater than one, then the generalised Lawton matrix has a real, symmetric eigenvalue of type (2) greater than one, i.e. has an eigenvalue of modulus greater than one. It follows that the matrix does not satisfy condition E. If eigenvalue σ is equal to one, then the generalised Lawton matrix has a real, symmetric eigenvalue of type (2) equal to one. Of course it also has a real, symmetric eigenvalue of type (1) equal to one. Hence the matrix has an eigenvalue at 1 of algebraic multiplicity greater than or equal to 2, i.e. the matrix does not satisfy condition E. If eigenvalue σ is less than one then, by Lemma 5.4, all of the symmetric eigenvalues of type (2) of the generalised Lawton matrix have modulus less than or equal to σ, i.e. less than one. By Lemma 5.5, $\rho = \sigma$, hence, by Lemma 5.3, the skew-symmetric eigenvalues of type (2) of the generalised Lawton matrix also have modulus less than one. The skew-symmetric eigenvalue of type (1) equals 1/2 and clearly has modulus less than one. Of course the symmetric eigenvalue of type (1) equals 1. Hence the matrix satisfies condition E. $\square$

An advantage of Theorem 5.2 is that it permits us to test condition E for a given generalised Lawton matrix, not by considering all of the eigenvalues of the matrix, but by considering the single eigenvalue σ which is known to be real, non-negative and symmetric of type (2). These known properties of σ simplify the numerical task of finding this eigenvalue.

6. The Lifting Scheme

We outline a single parameter form of the lifting scheme[8] as follows:

Theorem 6.1. *Take any initial pair of real, finite filters* $\{h, \tilde{h}\}$ *satisfying* $\hat{h}(0) = \hat{\tilde{h}}(0) = \sqrt{2}$, $\hat{h}(\pi) = \hat{\tilde{h}}(\pi) = 0$ *and*

$$\hat{\tilde{h}}(\zeta)\overline{\hat{h}(\zeta)} + \hat{\tilde{h}}(\zeta+\pi)\overline{\hat{h}(\zeta+\pi)} = 2 \tag{6.17}$$

which generate biorthogonal Riesz bases of compactly supported wavelets. Define two companion filters g *and* $\tilde{g}$ *as follows:*

$$\hat{g}(\zeta) = e^{-i\zeta}\overline{\hat{\tilde{h}}(\zeta+\pi)}\,, \quad \hat{\tilde{g}}(\zeta) = e^{-i\zeta}\overline{\hat{h}(\zeta+\pi)}\,, \tag{6.18}$$

then the new set $\{h, \tilde{h}^{\text{new}}\}$ *of finite filters which, together with their companion filters* $\{g^{\text{new}}, \tilde{g}\}$, *are generated as follows*:

$$\hat{\tilde{h}}^{\text{new}}(\zeta) = \hat{\tilde{h}}(\zeta) + \tau\hat{\tilde{g}}(\zeta)\overline{\hat{s}(2\zeta)}\,, \quad \hat{g}^{\text{new}}(\zeta) = \hat{g}(\zeta) - \tau\hat{h}(\zeta)\hat{s}(2\zeta)\,, \tag{6.19}$$

where $\hat{s}(\zeta)$ *is any real trigonometric,* 2π*-periodic polynomial and* τ *is any real parameter, obey the constraint of* (6.17).

This theorem is established by Sweldens.[8] Two filters, $\tilde{h}$ and g, are changed by the procedure which is known as the standard lifting scheme. A dual-lifting scheme involves replacing filters h and $\tilde{g}$. For the sake of brevity we restrict our discussion to the standard lifting scheme. Sweldens[8] effectively raises the question of whether one can determine simple conditions on $\hat{s}$ and τ such that the new filters can be guaranteed to generate biorthogonal Riesz bases of compactly supported wavelets. The key contribution of the present document will be to the effect that, for given $\hat{s}$, we can indeed find simple conditions on τ to guarantee generation of such bases. This is of course only a step towards the resolution Sweldens' question, not a complete resolution.

A simple necessary condition[2] for the new filters to generate the required bases is that $\hat{\tilde{h}}^{\text{new}}(0) = \sqrt{2}$ and $\hat{\tilde{h}}^{\text{new}}(\pi) = 0$. The former follows trivially from the nature of the lifting scheme itself, whereas it is apparent that the latter holds if and only if $\hat{s}(0) = 0$.

Lemma 6.1. *Assuming* $\hat{s}(0) = 0$ *the real filters* $\{h, \tilde{h}^{\mathrm{new}}\}$ *provided by the lifting scheme generate biorthogonal Riesz bases of compactly supported wavelets for all real* τ *in an open interval containing* 0. *Moreover, this interval is fully characterised by the facts that* (i) *it is maximal and* (ii) *at any boundary points of this interval the generalised Lawton matrix associated with the filter* $\tilde{h}^{\mathrm{new}}$ *has a symmetric eigenvalue of type* (2) *which is equal to* 1.

Proof. The new filters inherit many of the properties required to ensure existence of the requisite bases from the original filters and from the nature of the lifting scheme. In fact the only outstanding property is that the generalised Lawton matrix associated with $\tilde{h}^{\mathrm{new}}$ must satisfy condition E. Clearly $\tilde{h}^{\mathrm{new}}$ and its associated generalised Lawton matrix depend continuously upon parameter τ. The generalised Lawton matrix satisfies condition E for parameter value $\tau = 0$ (i.e. for the original filters) by the assumption of the lifting scheme. The requirement that the generalised Lawton matrix possesses an eigenvalue at 1 is guaranteed for arbitrary τ by the fact that $\hat{\tilde{h}}^{\mathrm{new}}(0) = \sqrt{2}$ and $\hat{\tilde{h}}^{\mathrm{new}}(\pi) = 0$ (which in turn is guaranteed by the fact that $\hat{s}(0) = 0$). It is elementary to establish that the set of all τ for which the generalised Lawton matrix satisfies condition E is open. Moreover, 0 is in this set, by assumption. It follows that this set must include an open interval containing 0 and must exclude any boundary points of this interval. From Theorem 5.2 it follows that the particular eigenvalue σ of the τ-dependent matrix must be greater than or to equal to 1 at these boundary points, but less than 1 for values of τ in the open interval itself, and as such, arbitrarily close to them. In consequence the eigenvalue σ must actually equal 1 at these boundary points. □

By reference to Lemmas 3.1 and 3.2 it is clear that the generalised Lawton matrix associated with $\tilde{h}^{\mathrm{new}}$ has a symmetric eigenvalue of type (2) equal to 1 if and only if $\det(I_{2M} - (A + BJ_{2M} - 2a[1, \ldots, 1])) = 0$. It is elementary to show that the coefficients of the matrix $(I_{2M} - (A + BJ_{2M} - 2a[1, \ldots, 1]))$ are quadratic polynomials in the variable τ. The evaluation of values of τ for which the determinant equals zero is therefore a special case of the quadratic eigenvalue problem.[4] By means of the method of linearisation[4] this problem may in general be converted to the problem of determining the eigenvalues of a larger matrix. Specifically, let

$$(I_{2M} - (A + BJ_{2M} - 2a[1, \ldots, 1])) = I_{2M} - C_0 - \tau C_1 - \tau^2 C_2 \tag{6.20}$$

for suitable constant, real matrices C_0, C_1, C_2. Assuming $(I_{2M} - C_0)$ is nonsingular, $\det(I_{2M} - (A + BJ_{2M} - 2a[1, \ldots, 1])) = 0$ for nonzero parameter value τ if and only if $(1/\tau)$ is an eigenvalue of the matrix

$$Q = \begin{pmatrix} 0 & I_{2M} \\ C_2(I_{2M} - C_0)^{-1} & C_1(I_{2M} - C_0)^{-1} \end{pmatrix} . \tag{6.21}$$

Employing this observation we obtain a corollary to Lemma 6.1 comprising a more readily tested stability condition.

Corollary 6.1. *Assuming that* $\hat{s}(0) = 0$ *and that* $(I_{2M} - C_0)$ *is non-singular, the real filters* $\{h, \tilde{h}^{\text{new}})$ *yielded by the lifting scheme generate biorthogonal Riesz bases of compactly supported wavelets for all real* τ *in an open interval containing* 0. *Moreover, the upper bound of this interval (if it exists) equals the reciprocal of the real, positive eigenvalue of* Q *of greatest modulus and the lower bound (if it exists) equals the reciprocal of the real, negative eigenvalue of* Q *of greatest modulus.*

Although this corollary calls for the inversion of matrix $(I_{2M} - C_0)$ and the determination of the eigenvalues of the potentially large matrix Q, numerical implementation is facilitated by two observations: (i) $(I_{2M} - C_0)$ is commonly highly structured so that its inversion requires relatively little effort, (ii) one does not seek all of the eigenvalues of matrix Q, but rather the largest real positive and largest real negative eigenvalues only.

7. An Example

As a simple illustration of the previous results, we commence with the Haar filters:

$$h = \tilde{h} = \sqrt{2}\left[0\, \frac{1}{2}\, \frac{1}{2}\right],$$

$$g = \tilde{g} = \sqrt{2}\left[0 - \frac{1}{2}\, \frac{1}{2}\right]. \tag{7.22}$$

It is readily shown that the Haar filters h, $\tilde{h}$ satisfy condition (6.17). Moreover, it is clear that the Haar filters g and $\tilde{g}$ are related to filters h, $\tilde{h}$ after the fashion of (6.18). We note also that $\hat{h}(0) = \hat{\tilde{h}}(0) = \sqrt{2}$, $\hat{h}(\pi) = \hat{\tilde{h}}(\pi) = 0$ The generalised Lawton matrix for both filters h and $\tilde{h}$ is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.23}$$

It satisfies the column sum condition and has eigenvalues $0, 0, 1, \frac{1}{2}, \frac{1}{2}$. Clearly the matrix satisfies condition E. We perform a standard lifting step on these filters using the fixed real trigonometric polynomial:

$$\hat{s}(\zeta) = (-e^{i\zeta} + e^{-}i\zeta) \tag{7.24}$$

which clearly satisfies $\hat{s}(0) = 0$. The new filters are readily found from (6.19):

$$h = \sqrt{2}\left[0\,0\,0\,\frac{1}{2}\,\frac{1}{2}\,0\,0\right], \quad \tilde{g} = \sqrt{2}\left[0\,0\,0\,-\frac{1}{2}\,\frac{1}{2}\,0\,0\right],$$

$$\tilde{h}^{\text{new}} = \sqrt{2}\left[0 - \frac{\tau}{2}\,\frac{\tau}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{\tau}{2} - \frac{\tau}{2}\right],$$

$$g^{\text{new}} = \sqrt{2}\left[0\,\frac{\tau}{2}\,\frac{\tau}{2} - \frac{1}{2}\,\frac{1}{2} - \frac{\tau}{2} - \frac{\tau}{2}\right]. \tag{7.25}$$

It is the nature of the lifting scheme (when $\hat{s}(0) = 0$) that satisfaction of all of the requirements of the stability condition of Theorem 2.1 is inherited from the starting filters h and $\tilde{h}$ except for the single requirement that the generalised Lawton matrix associated with filter $\tilde{h}^{\text{new}}$ satisfy condition E. Consequently we focus our attention on this matrix, $\tilde{\Lambda}^{\text{new}}$, which, by construction, depends on the real parameter τ, is real and satisfies the column sum condition for all τ, and which satisfies condition E when $\tau = 0$. As the matrix is 13×13 we elect not to write it out in full. However, by comparing with the canonical structure of Lemma 4.2 we identify the sub-matrices:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 \\ 0 & -\tau + \frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 \\ 1 + 2\tau^2 & \frac{1}{2} + \tau - \tau^2 & 0 & -\tau + \frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} \\ 0 & \frac{1}{2} + \tau - \tau^2 & 1 + 2\tau^2 & \frac{1}{2} + \tau - \tau^2 & 0 & -\tau + \frac{\tau^2}{2} \\ -\tau^2 & -\tau + \frac{\tau^2}{2} & 0 & \frac{1}{2} + \tau - \tau^2 & 1 + 2\tau^2 & \frac{1}{2} + \tau - \tau^2 \end{pmatrix}, \quad a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\tau^2 \\ 0 \end{pmatrix}, \tag{7.26}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 \\ \left(-\tau + \frac{\tau^2}{2}\right) & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \frac{\tau^2}{2} \\ -\tau^2 \\ -\tau + \frac{\tau^2}{2} \\ 0 \\ \frac{1}{2} + \tau - \frac{\tau^2}{2} \end{pmatrix}, \tag{7.27}$$

$$c = 1 + 2\tau^2. \tag{7.28}$$

The symmetric eigenvalues of type (2) are the eigenvalues of the reduced order matrix

$$(A + BJ_{2M} - 2a[1,\dots,1])$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 \\ 0 & \left(-\tau + \frac{\tau^2}{2}\right) & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 \\ (1+2\tau^2) & \left(\frac{1}{2} + \tau - \tau^2\right) & 0 & \left(-\tau + \frac{\tau^2}{2}\right) & -\tau^2 & \frac{\tau^2}{2} \\ 2\tau^2 & \left(\frac{1}{2} + \tau - \tau^2\right) & (1+4\tau^2) & \left(\frac{1}{2} + \tau + \tau^2\right) & 2\tau^2 & (-\tau + 3\tau^2) \\ -\tau^2 & \left(-\tau + \frac{\tau^2}{2}\right) & 0 & \left(\frac{1}{2} + \tau - \frac{\tau^2}{2}\right) & (1+\tau^2) & \left(\frac{1}{2} - \frac{\tau^2}{2}\right) \end{pmatrix}. \tag{7.29}$$

We note in this case that

$$I_{2M} - C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -\frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}. \tag{7.30}$$

Not only is this matrix non-singular, it is also lower triangular and therefore readily invertible. It is feasible to construct Q and determine its eigenvalues. In this case, however, we do not actually need to employ linearisation, since one may readily show that $\det(I_{2M} - (A + BJ_{2M} - 2a[1,\dots,1]))$ equals the polynomial:

$$\frac{1}{2}\left(1 - \frac{\tau^2}{2}\right)(1+\tau^2)(1+2\tau-8\tau^2)(1+\tau)\,, \tag{7.31}$$

whose roots are: $\pm\sqrt{2}$, $\pm i, -1$, $-\frac{1}{4}$, $\frac{1}{2}$. The maximal real open interval containing 0 with boundary points, but no interior points, in this set is given by $-\frac{1}{4} < \tau < \frac{1}{2}$. Hence, for any value of τ between $-\frac{1}{4}$ and $\frac{1}{2}$ the resulting filters $(h, \tilde{h}^{\text{new}})$ generate biorthogonal Riesz bases of compactly supported wavelets. Note that the method does not generate a single set of new filters, but rather an entire class of new filters parametrised by the single parameter τ.

For other starting filters $\{h, \tilde{h}\}$ and other lifting filters s the matrices C_0, C_1 and C_2 can be systematically generated but will tend to become quite large. It is then appropriate to resort to a numerical method to generate the matrix Q and to determine its relevant eigenvalues. For example, with

$$h = \sqrt{2}\left[\frac{1}{4}\,\frac{1}{2}\,\frac{1}{4}\right]\,, \quad \tilde{h} = \sqrt{2}\left[-\frac{1}{2}\,\frac{1}{4}\,\frac{3}{4}\,\frac{1}{4}\,-\frac{1}{2}\right]\,, \quad s = [-1\,0\,1]\,, \tag{7.32}$$

one obtains

$$\tilde{h}^{\mathrm{new}} = \sqrt{2}\left[0, 0, -\frac{1}{8} - \frac{\tau}{4}, \frac{1}{4} + \frac{\tau}{2}, \frac{3}{4} - \frac{\tau}{4}, \frac{1}{4}, -\frac{1}{8} + \frac{\tau}{4}, -\frac{\tau}{2}, \frac{\tau}{4}\right] \tag{7.33}$$

and concludes that the new filters $\{h, \tilde{h}^{\mathrm{new}}\}$ generate biorthogonal Riesz bases of compactly supported wavelets for $-0.1756 < \tau < 0.3392$.

8. Conclusion

We have presented a new stability criterion for biorthogonal wavelet bases with an associated subband coding scheme. It has been established that, given a generalised Lawton matrix satisfying the column sum condition, there exists a particular real positive eigenvalue of this matrix, σ, which is not less in modulus than the remaining eigenvalues (excluding the special eigenvalues at one and a half). Stability is maintained as long as σ is less than one. It has been demonstrated that this stability criterion, in conjunction with a single-parameter formulation of the lifting scheme, can be used to design biorthogonal filter banks with associated wavelet bases in $L_2(R)$. Numerically the new stability test is a special case of the quadratic eigenvalue problem. For low order filters the method of linearisation is appropriate and reduces the problem to a standard eigenvalue problem (or rather to the problem of finding the largest and smallest nonzero real eigenvalues of a matrix). A single matrix inversion is required in the application of the linearisation method. However, the matrix to be inverted often has a structure which facilitates rapid inversion. For higher order filters more advanced techniques for solving the quadratic eigenvalue problem (e.g. the Jacobi–Davidson method) would be required.

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