Single Parameter Lifting Scheme for Custom Design of Wavelets in $L_2(R)$.

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Abstract: **For the case of real finite filters we investigate a parameterised form of the lifting scheme containing a single parameter. We show that the scheme generates biorthogonal filter banks having associated wavelets in $L_2(R)$ provided the parameter lies in an open interval containing zero and develop an algorithm for finding the largest such interval. In conjunction with the lifting scheme, the algorithm is used to custom design parameterised classes of biorthogonal filter banks with associated wavelets in $L_2(R)$.**

1. Introduction

In 1988 Daubechies [3] discovered a class of compactly supported orthonormal bases for $L_2(R)$ which included the Haar basis as a special case. Mallat [5] established the relationship between wavelet transforms and multi-resolution analyses and showed that a discrete wavelet transform (relative to an orthonormal basis) can be implemented using orthogonal filter bank theory.

Whereas orthogonality of the basis is a useful property in the analysis/synthesis of signals, it is not indispensable. In 1992 Cohen, Daubechies and Feauveau [2] introduced the idea of biorthogonal wavelet bases. In this case two distinct bases are employed, one for analysis and one for synthesis. The two bases are not necessarily orthogonal in their own right but are orthogonal to one another. Biorthogonal bases offer increased flexibility in the design of the associated filter bank enabling for example, the construction of filter banks from linear phase filters. Cohen, Daubechies and Feauveau [2], Cohen and Daubechies [1] and Strang [7] provide necessary and sufficient conditions for a pair of dual filters to generate biorthogonal compactly supported wavelet bases in $L_2(R)$.

Sweldens [8] introduced the lifting scheme for designing biorthogonal filter banks. This scheme formally maintains biorthogonality but does not guarantee that the filter bank has associated compactly supported wavelet bases in $L_2(R)$. Whereas the lifting scheme in general contains many free parameters we reformulate it below in terms of a single parameter. For real, finite filters the single parameter dependent lifting scheme generates biorthogonal filter banks having associated wavelets in $L_2(R)$ provided the parameter lies in an open interval containing zero. We present an algorithm for finding the largest interval of this kind. In conjunction with the lifting scheme, this algorithm provides a method for the custom design of biorthogonal filter banks with associated wavelet bases in $L_2(R)$. While the method is cumbersome for large filters it has been found to be numerically tractable for filters having up to twenty taps.

2. Lawton Matrices

Given $h = [h_{-m}, \ldots, h_{-1}, h_0, h_1, \ldots, h_m]$, a real filter of length $(2m+1)$, as usual we define the *Z-transform* of the filter to be:

$$H(z) = \sum_{k=-m}^{m} h_k z^{-k}.$$

We say that filter *h* is *balanced* if $H(1) = 1$. We define also an associated real sequence η by $\eta_k = 2\sum_q h_{q+k} h_q$ where the filter coefficients with indices outside range $-m$ to m are defined to be zero. The real Lawton matrix [4] Λ associated with the filter is the $(4m+1)\times(4m+1)$ matrix:

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$$\Lambda = \begin{bmatrix} \eta_{2m} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \eta_{2m-2} & \eta_{2m-1} & \eta_{2m} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \eta_{2m} & \eta_{2m-1} & \eta_{2m-2} & \eta_{2m-2} & \cdots & \eta_2 & \eta_1 & \eta_0 & \eta_1 & \cdots & \eta_{2m-3} & \eta_{2m-2} & \eta_{2m-1} & \eta_{2m} \\ 0 & 0 & \eta_{2m} & \eta_{2m-1} & \cdots & \eta_4 & \eta_3 & \eta_2 & \eta_1 & \cdots & \eta_{2m-5} & \eta_{2m-4} & \eta_{2m-3} & \eta_{2m-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \eta_{2m} & \eta_{2m-1} & \eta_{2m-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \eta_{2m} \end{bmatrix}$$

We define a pair of *dual real finite filters* to be a set of two real balanced filters $\left(h, \tilde{h}\right)$ of length $(2m+1)$ and $(2\tilde{m}+1)$ respectively such that

$$\tilde{H}\left(e^{i\theta}\right)\overline{H\left(e^{i\theta}\right)} + \tilde{H}\left(e^{i(\theta+\pi)}\right)\overline{H\left(e^{i(\theta+\pi)}\right)} = 1$$

for all θ. The following result is well-known:

Lemma 1: A pair of dual real finite filters, $\left(h, \tilde{h}\right)$, generate biorthogonal Riesz bases of compactly supported wavelets only if the sum of the elements in every column of the Lawton matrix associated with each of the filters is one.

We call this necessary condition on the Lawton matrix associated with a balanced real filter the *column sum condition*. It transpires that the column sum condition corresponds to a simple condition on the filter itself:

Lemma 2: The Lawton matrix associated with a real balanced filter h of length $(2m+1)$ satisfies the column sum condition iff $H(-1) = 0$.

3. Lawton Symmetry

Given a matrix $A \in C^{N\times M}$ with coefficients a_{ij} let matrix $A' \in C^{N\times M}$ be defined by:

$$[A']_{ij} = \bar{a}_{N+1-i,\ M+1-j}$$

for all indices i and j.

Subject to this definition a matrix A is said to be *Lawton symmetric* if $A = A'$. It is not difficult to show that the Lawton matrix associated with a real filter of length $(2m+1)$ is real and Lawton symmetric. We observe also the following result whose proof is elementary:

Lemma 3: A real, $(2M+1)\times(2M+1)$, Lawton symmetric matrix, Λ, has the following structure:

$$\Lambda = \begin{bmatrix} A & a & B \\ b^T & c & (b')^T \\ B' & a' & A' \end{bmatrix}$$

where $A, B \in R^{M\times M}$, $a, b \in R^M$ and $c \in R$.

Concerning real, $(2M+1)\times(2M+1)$, Lawton symmetric matrices satisfying the column sum condition the following result proves to be of importance:

Lemma 4: M of the eigenvalues of a real, $(2M+1)\times(2M+1)$, Lawton symmetric matrix Λ satisfying the column sum condition may be classified are eigenvalues of the reduced order matrix $\left(A + BE - 2aw^T\right)$, $w^T = [1, \ldots, 1]$.

We call these the *symmetric eigenvalues of type (2)*.

4. The Lifting Scheme

We outline a paramaterised lifting scheme as follows:

Theorem 1: Take any initial set of real finite, balanced filters $\{h, \tilde{h}\}$ satisfying the biorthogonal constraint:

$$\tilde{H}\left(e^{i\theta}\right)\overline{H\left(e^{i\theta}\right)} + \tilde{H}\left(e^{i(\theta+\pi)}\right)\overline{H\left(e^{i(\theta+\pi)}\right)} = 1.$$

Assume that these filters generate biorthogonal Riesz bases of compactly supported wavelets Define companion filters g and $\tilde{g}$ as follows:

$$\tilde{G}\left(e^{i\theta}\right) = e^{-i\theta}\,\overline{H\left(e^{i(\theta+\pi)}\right)}$$

$$G\left(e^{i\theta}\right) = e^{-i\theta}\,\overline{\tilde{H}\left(e^{i(\theta+\pi)}\right)}$$

then a new set of finite balanced filters $\{h, \tilde{h}^{new}\}$, together with their companion filters $\{g^{new}, \tilde{g}\}$, are generated as follows:

$$\tilde{H}^{new}\left(e^{i\theta}\right) = \tilde{H}\left(e^{i\theta}\right) + \tau\,\tilde{G}\left(e^{i\theta}\right)\overline{S\left(e^{i2\theta}\right)}$$

$$G^{new}\left(e^{i\theta}\right) = G\left(e^{i\theta}\right) - \tau\,H\left(e^{i\theta}\right)S\left(e^{i2\theta}\right)$$

where $S\left(e^{i\theta}\right)$ is a real trigonometric polynomial and τ is a real parameter. These new filters also satisfy the biorthogonal constraint.

The question arises as to whether, for a given real trigonometric polynomial S and real parameter τ the filters $\left\{ h, \tilde{h}^{new} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets. A simple necessary condition [6] is stated as follows:

Lemma 5: The filters $\left\{ h, \tilde{h}^{new} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets only if $S(1) = 0$.

Theorem 2: Assuming $S(1) = 0$ the filters $\left\{ h, \tilde{h}^{new} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets for all real τ in an open interval containing 0. Moreover, this interval is characterised by the facts that it is maximal and that at any boundary points, but at no interior points, the Lawton matrix associated with $\tilde{h}^{new}$ has a symmetric eigenvalue of type (2) equal to 1.

5. An Example

Select initially Haar filters $h = \left[0, \frac{1}{2}, \frac{1}{2}\right] = \tilde{h}$ and their companion filters $g = \left[0, -\frac{1}{2}, \frac{1}{2}\right] = \tilde{g}$. It is readily shown that filters h, $\tilde{h}$ satisfy the biorthogonal constraint. Note that filters h and $\tilde{h}$ are real, finite and balanced and that $H(-1) = \tilde{H}(-1) = 0$. They comprise a dual real finite pair of filters. The Lawton matrix associated with both filters is:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It satisfies the column sum condition and has eigenvalues $0, 0, 1, \frac{1}{2}, \frac{1}{2}$. One eigenvalue is 1. It is simple and strictly exceeds all other eigenvalues in modulus. It follows from [2] that filters $\left\{ h, \tilde{h} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets. We apply the parameterised lifting scheme using the fixed real trigonometric polynomial:

$$S\left(e^{i\theta}\right) = \left(-e^{i\theta} + e^{-i\theta}\right)$$

which clearly satisfies $S(1) = 0$. The new filters are

$$\begin{aligned} h &= \left[0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right] \\ \tilde{g} &= \left[0, 0, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right] \\ \tilde{h}^{new} &= \left[0, -\tfrac{\tau}{2}, \tfrac{\tau}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{\tau}{2}, -\tfrac{\tau}{2}\right] \\ g^{new} &= \left[0, \tfrac{\tau}{2}, \tfrac{\tau}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{\tau}{2}, -\tfrac{\tau}{2}\right] \quad . \end{aligned}$$

The Lawton matrix associated with filter $\tilde{h}^{new}$, denoted Λ, is given as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 \\ -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 \\ 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 \\ 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 \\ 0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is Lawton symmetric. By comparing with the canonical structure of lemma 3 we identify the sub- matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 \\ 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 \\ 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} \\ 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} \\ -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\tau^2 \\ 0 \end{bmatrix}$$

$$
B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 \\ -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \frac{\tau^2}{2} \\ -\tau^2 \\ -\tau+\frac{\tau^2}{2} \\ 0 \\ \frac{1}{2}+\tau-\frac{\tau^2}{2} \end{bmatrix}
$$

$$
c = 1 + 2\tau^2 .
$$

The symmetric eigenvalues of type (2) are the eigenvalues of the reduced order matrix $\left(A + BE - 2aw^T\right) =$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 \\ 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 \\ 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} \\ 2\tau^2 & \frac{1}{2}+\tau+\tau^2 & 1+4\tau^2 & \frac{1}{2}+\tau+\tau^2 & 2\tau^2 & -\tau+3\tau^2 \\ -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\frac{\tau^2}{2} & 1+\tau^2 & \frac{1}{2}-\frac{\tau^2}{2} \end{bmatrix}.
$$

The matrix has a symmetric eigenvalue of type (2) equal to 1 iff $\det\left(I - \left(A + BE - 2aw^T\right)\right) = 0$ where I is the identity matrix. In the present case this determinant equals the polynomial in τ:

$$
\frac{1}{2}\left(1 - \frac{\tau^2}{2}\right)\left(1+\tau^2\right)\left(1+2\tau-8\tau^2\right)\left(1+\tau\right)
$$

whose roots are: $\pm\sqrt{2}, \pm i, -1, -\frac{1}{4}, \frac{1}{2}$. The maximal real open interval containing 0 with boundary points, but no interior points, in this set is given by $-\frac{1}{4} < \tau < \frac{1}{2}$. Hence, for any value of τ between $-1/4$ and $1/2$ the resulting filters $\left\{ h, \tilde{h}^{new} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets.

6. Conclusions

We have formulated a parameterised lifting scheme with a single real parameter. We have shown that the scheme generates biorthogonal filter banks having associated wavelets in $L_2(R)$ provided the parameter lies in a certain open interval and have developed a method for finding the largest such interval. We note that the parameterised lifting scheme, in conjunction with this method, yields a whole class of biorthogonal filter banks with associated wavelet bases in $L_2(R)$ and that this class is itself parameterised. Clearly one may employ a stochastic algorithm to determine the filter bank in this parameterised class which is optimal with respect to some desirable property (such as maximum energy compaction, desired shape, *etc*.).

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