Custom Design of Wavelets in $L_2(R)$.

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Abstract

The existence in $L_2(R)$ of biorthogonal wavelet bases associated with perfect reconstruction 2 channel filter banks comprising finite, real filters is determined by the eigenvalues of two related Lawton matrices. Each Lawton matrix is required to have a simple eigenvalue at one and all remaining eigenvalues must have modulus less than one. This being the case then the wavelet bases are said to be stable. If the filter banks are changed in any way then the eigenvalues of the associated Lawton matrices must be re-calculated to determine the stability of the new wavelet bases. This is a numerically intensive task. In this paper we present a new stability criterion that is far simpler to test. Starting with stable biorthogonal wavelet bases we perturb the associated filter banks, preserving perfect reconstruction and ensuring that the new Lawton matrices continue to have an eigenvalue at one. We show that stability of the new biorthogonal wavelet bases first breaks down, not just when a second eigenvalue attains a modulus of one, but rather when a second eigenvalue actually equals one. This leads to the numerically simpler stability criterion of counting eigenvalues at one in the Lawton matrices. The new criterion is used in conjunction with the lifting scheme to construct a perfect reconstruction filter bank with associated biorthogonal wavelet bases. This provides a general-purpose algorithm for the custom design of filter banks that are guaranteed to be stable.

1. INTRODUCTION

In 1988 Daubechies [1] discovered a class of compactly supported orthonormal bases for $L_2(R)$ which included the Haar basis as a special case. Mallat [2] established the relationship between wavelet transforms and multi-resolution analyses and showed that a discrete wavelet transform (relative to an orthonormal basis) can be implemented using orthogonal filter bank theory.

Whereas orthogonality of the basis is a useful property in the analysis/synthesis of signals, it is not indispensable. In 1992 Cohen, Daubechies and Feauveau [3] introduced the idea of biorthogonal wavelet bases. In this case two distinct bases are employed, one for analysis and one for synthesis. The two bases are not necessarily orthogonal in their own right but are orthogonal to one another. Biorthogonal bases offer increased flexibility in the design of the associated filter bank enabling, for example, the construction of filter banks from linear phase filters. Cohen, Daubechies and Feauveau [3], Cohen and Daubechies [4] and Strang [5] provide necessary and sufficient conditions for a pair of dual filters to generate biorthogonal compactly supported wavelet bases in $L_2(R)$.

Sweldens [6] introduced the lifting scheme for designing biorthogonal filter banks. This scheme formally maintains biorthogonality but does not guarantee that the filter bank has associated compactly supported wavelet bases in $L_2(R)$. Whereas the lifting scheme in general contains many free parameters we reformulate it in terms of a single parameter. The principle contribution of the present work is the observation that for real, finite filters the single parameter dependent lifting scheme generates biorthogonal filter banks having associated wavelets in $L_2(R)$ provided the parameter lies in an open interval containing zero. We present an algorithm for finding the largest interval of this kind. In conjunction with the lifting scheme, this algorithm provides a method for the custom design of biorthogonal filter banks with associated wavelet bases in $L_2(R)$. While the method is cumbersome for large filters it has been found to be numerically tractable for filters having up to twenty taps. The resulting wavelet bases depend continuously upon the single parameter of the lifting scheme. It is therefore possible, by employing a variety of different optimisation techniques, to select the value of this parameter such that the associated wavelet basis is optimal in some sense.

2. LAWTON MATRICES

Given $h = [h_{-m}, \ldots, h_{-1}, h_0, h_1, \ldots, h_m]$, a real filter of length $(2m+1)$, as usual we define the *Z-transform* of the filter to be:

$$H(z) = \sum_{k=-m}^{m} h_k \, z^{-k} \, .$$

We say that filter *h* is *balanced* if $H(1) = 1$. We define also an associated real sequence η by $\eta_k = 2\sum_q h_{q+k} h_q$ where the filter coefficients with indices outside range $-m$ to m are defined to be zero. The real Lawton matrix [7] Λ associated with the filter is the $(4m+1) \times (4m+1)$ matrix:

$$\Lambda = \begin{bmatrix}
\eta_{2m} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\eta_{2m-2} & \eta_{2m-1} & \eta_{2m} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\eta_{2m} & \eta_{2m-1} & \eta_{2m-2} & \eta_{2m-2} & \cdots & \eta_2 & \eta_1 & \eta_0 & \eta_1 & \cdots & \eta_{2m-3} & \eta_{2m-2} & \eta_{2m-1} & \eta_{2m} \\
0 & 0 & \eta_{2m} & \eta_{2m-1} & \cdots & \eta_4 & \eta_3 & \eta_2 & \eta_1 & \cdots & \eta_{2m-5} & \eta_{2m-4} & \eta_{2m-3} & \eta_{2m-2} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \eta_{2m} & \eta_{2m-1} & \eta_{2m-2} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \eta_{2m}
\end{bmatrix}$$

We define a pair of *dual real finite filters* to be a set of two real balanced filters $(h, \tilde{h})$ of length $(2m+1)$ and $(2\tilde{m}+1)$ respectively such that

$$\tilde{H}(e^{i\theta})\overline{H(e^{i\theta})} + \tilde{H}(e^{i(\theta+\pi)})\overline{H(e^{i(\theta+\pi)})} = 1 \text{ for all } \theta.$$

The following result is well known [4]:

Stability Condition: A pair of dual real finite filters, $(h, \tilde{h})$, generate biorthogonal Riesz bases of compactly supported wavelets iff the Lawton matrix associated with each filter has a simple eigenvalue at one and all remaining eigenvalues have modulus less than one.

An elementary corollary may be stated as follows:

Lemma 1: A pair of dual real finite filters, $(h, \tilde{h})$, generate biorthogonal Riesz bases of compactly supported wavelets only if the sum of the elements in every column of the Lawton matrix associated with each of the filters is one.

We call this necessary condition on the Lawton matrix associated with a balanced real filter the *column sum condition*. It transpires that the column sum condition corresponds to a simple condition on the filter itself [8]:

Lemma 2: The Lawton matrix associated with a real balanced filter *h* of length $(2m+1)$ satisfies the column sum condition iff $H(-1)=0$.

3. LAWTON SYMMETRY

Given a matrix $A \in C^{N\times M}$ with coefficients a_{ij} let matrix $A' \in C^{N\times M}$ be defined by:

$$[A']_{ij} = \bar{a}_{N+1-i,\ M+1-j}$$

for all indices *i* and *j*.

Subject to this definition a matrix *A* is said to be *Lawton symmetric* if $A = A'$. It is not difficult to show that the Lawton matrix associated with a real filter of length $(2m+1)$ is real and Lawton symmetric. We observe also the following result:

Lemma 3: A real, $(2M+1)\times(2M+1)$, Lawton symmetric matrix, Λ, has the following structure:

$$\Lambda = \begin{bmatrix} A & a & B \\ b^T & c & (b')^T \\ B' & a' & A' \end{bmatrix}$$

where $A, B \in R^{M\times M}$, $a, b \in R^M$ and $c \in R$.

Let

$$E = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \in R^{M\times M}$$

Concerning real, $(2M+1)\times(2M+1)$, Lawton symmetric matrices satisfying the column sum condition the following result proves to be of importance:

Lemma 4: The eigenvalues of a real, $(2M+1)\times(2M+1)$, Lawton symmetric matrix Λ satisfying the column sum condition may be classified as follows:

1. One of them equals 1.
2. A further M of them are the eigenvalues of the reduced order matrix $(A - BE)$ with one of these being ½.
3. The remaining M are eigenvalues of the reduced order matrix $(A + BE - 2aw^T)$, $w^T = [1, \ldots, 1]$.

We call the eigenvalue at 1 the *symmetric eigenvalue of type (1)* and that at ½ the *skew-symmetric eigenvalue of type (1).* The remaining M-1 eigenvalues of the second class we call the *skew-symmetric eigenvalues of type (2)* and the eigenvalues of the third class we call the *symmetric eigenvalues of type (2).*

The terminology is inspired by the fact that symmetric eigenvalues have associated eigenvectors which are *symmetric* i.e. have the following form:

$$\begin{bmatrix} x_1 \\ x_0 \\ Ex_1 \end{bmatrix} \qquad \text{for some complex, } M \times 1 \text{ vector } x_1 \text{ and complex number } x_0 .$$

Similarly skew-symmetric eigenvalues have associated eigenvectors which are *skew-symmetric* i.e. which have the following form:

$$\begin{bmatrix} y_1 \\ 0 \\ -Ey_1 \end{bmatrix} \qquad \text{for some complex, } M \times 1 \text{ vector } y_1 .$$

4 NON-NEGATIVENESS

Given any vector $v \in R^{2M+1}$ v will be said to be non-negative, denoted $v \geq 0$, if:

$$\mathrm{Re}\left(V\left(e^{i\theta}\right)\right) + \mathrm{Im}\left(V\left(e^{i\theta}\right)\right) \geq 0 \quad \forall\, \theta \in [0, 2\pi]$$

where $V(z)$ denotes the Z-transform of *v* as above. All subsequent references to non-negative vectors are in this sense.

Krein and Rutman [9] define a convex cone in a finite dimensional, real vector space to be a subset, *C*, of the vector space having the following properties:

1. If $x \in C$ then $\alpha x \in C$ for all scalars $\alpha \geq 0$.
2. If $x, y \in C$ then $x + y \in C$.
3. If $x, y \in C, x \neq 0$ then $x + y \neq 0$.
4. *C* is closed relative to the standard Euclidean norm-topology on the vector space.

Consider the set of all real, $(2M+1) \times 1$ non-negative vectors:

$$K = \left\{ v \in R^{2M+1} \mid v \geq 0 \right\}.$$

The set *K* has two properties that prove to be significant in the study of Lawton matrices:

Lemma 5: *K* is a convex cone (in the sense of Krein and Rutman).

Lemma 6: $K + (-K) = R^{2M+1}$.

The previous results permit a number of corollaries. Let $L = \left\{ v \in R^{2M+1} \mid v = v' \right\}$, i.e. *L* is the set of all real, Lawton symmetric, $(2M+1) \times 1$ vectors.

Corollary 1:

1. *L* is a subspace of R^{2M+1}.
2. A real Lawton symmetric matrix maps *L* into itself.

3. $K \cap L$ is a convex cone in *L*.
4. $(K \cap L) + (-K \cap L) = L$.

Let $Z_0 = \left\{ \; v \in R^{2M+1} \middle| \; [1, \ldots, 1] \; v = 0 \; , \; [M, \ldots, 1, 0, -1, \ldots, -M] \; v = 0 \; \right\}$.

Corollary 2:
1. Z_0 is a subspace of R^{2M+1}.
2. A real Lawton symmetric matrix that satisfies the column sum condition maps Z_0 into itself.
3. $K \cap Z_0$ is a convex cone in Z_0.
4. $(K \cap Z_0) + (-K \cap Z_0) = Z_0$.

Corollary 3:
1. $Z_0 \cap L$ is a subspace of R^{2M+1}.
2. A real Lawton symmetric matrix that satisfies the column sum condition maps $Z_0 \cap L$ into itself.
3. $K \cap Z_0 \cap L$ is a convex cone in $Z_0 \cap L$.
4. $(K \cap Z_0 \cap L) + (-K \cap Z_0 \cap L) = Z_0 \cap L$.

One further property of non-negative, real vectors that will be required in our subsequent discussion of Lawton matrices may be stated as follows:

Lemma 7: There exists no non-zero, real, skew-symmetric, non-negative vector in R^{2M+1}.

Corresponding to the definition of non-negative, real vectors given above we now propose the following definition of non-negative, real matrices:

Definition 6: Given any matrix $\Lambda \in R^{(2M+1)\times(2M+1)}$ we say that Λ is non-negative, denoted $\Lambda \geq 0$, if $\Lambda v \geq 0$ for all $v \geq 0$ in $R^{(2M+1)}$.

A significant feature of Lawton matrices associated with real, finite filters is that they are non-negative. This observation is formally stated as follows:

Lemma 8: The Lawton matrix, Λ, associated with a real filter of length $(2m+1)$ is non-negative.

In terms of the cones introduced previously lemma 8 asserts that a Lawton matrix associated with a real, finite filter of length $(2m+1)$ defines a linear operator on real vector space R^{2m+1} which maps the convex cone *K* into itself. There exist some elementary, but important, corollaries to this result:

Corollary 4: By restriction, a Lawton matrix associated with a real, finite filter defines linear operators on real vector spaces L, Z_0, $Z_0 \cap L$ which map the convex cones $K \cap L$, $K \cap Z_0$, $K \cap Z_0 \cap L$ respectively into themselves.

GENERALISED FROBENIUS-PERRON THEORY

In their celebrated treatise, Krein and Rutman [9] present a generalisation of the classical Frobenius-Perron theorem which we may paraphrase as follows:

Theorem 1: Let *K* be a convex cone with non-null interior in a real, finite-dimensional vector space; if a linear mapping Λ maps *K* into itself and is not nilpotent, then there is a real, positive eigenvalue λ_K of Λ with an associated eigenvector lying in *K*, having the property that no other eigenvalue of Λ has modulus exceeding λ_K .

By employing the results of section 4 together with theorem 1, we may make a number of assertions concerning Lawton matrices associated with real filters. We will employ also the elementary result that a square matrix of finite dimension, possessing a non-zero eigenvalue, is not nilpotent.

Lemma 9: Let Λ be a Lawton matrix associated with a real, finite, balanced filter which satisfies the column sum condition, then there exists a real, positive eigenvalue, L, of Λ such that:

(i) all remaining eigenvalues of Λ have modulus less than or equal to L,
(ii) there exists a real, non-negative eigenvector, $v_{(L)}$, associated with L.

Lemma 10: Let Λ be a Lawton matrix associated with a real, finite, balanced filter which satisfies the column sum condition, then there exists a real, positive symmetric eigenvalue, S, of Λ such that:

(i) all remaining symmetric eigenvalues of Λ have modulus less than or equal to S,
(ii) there exists a real, non-negative eigenvector, $v_{(S)}$, associated with S.

Lemma 11: Let Λ be a Lawton matrix associated with a real, finite, balanced filter which satisfies the column sum condition, then there exists a real, positive eigenvalue, ρ, of Λ which is either symmetric of type (2) or skew-symmetric of type (2) such that

(i) all remaining symmetric and skew-symmetric eigenvalues of type (2) of Λ have modulus less than or equal to ρ.
(ii) there exists a real, non-negative eigenvector, $v_{(\rho)}$, associated with ρ.

Lemma 12: Let Λ be a Lawton matrix associated with a real, finite, balanced filter which satisfies the column sum condition, then there exists a real, positive eigenvalue, σ, of Λ which is symmetric of type (2) such that:

(i) all remaining symmetric eigenvalues of type (2) of Λ have modulus less than or equal to σ,
(ii) there exists a non-negative eigenvector, $v_{(\sigma)}$, associated with σ.

Lemma 8 permits us to make a number of observations concerning the eigenvalues L, S, ρ and σ of lemmas 9-12. The proof of these observations is included to indicate the utility of lemma 8.

Lemma 13: Eigenvalue L is symmetric and equals eigenvalue S. Eigenvalue ρ is symmetric of type (2) and equals eigenvalue σ.

Proof: Lemma 9 assures that $v_{(L)}$ is real, non-zero and non-negative. By lemma 8 this vector cannot, therefore, be skew-symmetric. Hence eigenvalue L cannot be skew-symmetric and must, therefore, be symmetric. It is now trivial to show that L = S.

Lemma 11 assures that $v_{(\rho)}$ is real, non-zero and non-negative. As above, lemma 8 asserts that this vector cannot be skew-symmetric and, therefore, that eigenvalue ρ cannot be skew-symmetric. Hence ρ must be symmetric of type (2) and it is now trivial to show that $\rho = \sigma$.

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Note that the eigenvalue σ, of lemma 12, is uniquely defined by the Lawton matrix (and hence by the real filter associated with it). We are finally in a position to state and prove the primary result of this investigation:

Theorem 2: The Lawton matrix associated with a real, finite, balanced filter satisfying $H(-1) = 0$ has a simple eigenvalue at one and all remaining eigenvalues have modulus less than one if and only if the particular eigenvalue σ is less than 1.

Proof: The conditions imposed imply that the associated Lawton matrix is real and satisfies the column sum condition. Hence, the division of eigenvalues into symmetric eigenvalues of types (1) and (2) and skew-symmetric eigenvalues of types (1) and (2) is valid.

If σ is greater than 1 then the Lawton matrix has a real, symmetric eigenvalue of type (2) greater than 1. It follows that the Lawton matrix does not satisfy the eigenvalue condition.

If σ is equal to 1 then the Lawton matrix has a real, symmetric eigenvalue of type (2) equal to 1. Of course it also has a real, symmetric eigenvalue of type (1) equal to 1. Hence the matrix has an eigenvalue at 1 of algebraic multiplicity greater than or equal to 2. It follows the Lawton matrix does not satisfy the eigenvalue condition.

If σ is less than 1 then, by lemma 12, all of the symmetric eigenvalues of type (2) of the Lawton matrix have modulus less than or equal to σ, i.e. less than 1. By lemma 13, $\rho = \sigma$, hence, by lemma 11, the skew-symmetric eigenvalues of type (2) of the associated Lawton matrix also have modulus less than 1. The skew-symmetric eigenvalue of type (1) equals ½ and clearly has modulus less than 1. Of course the symmetric eigenvalue of type (1) equals 1. Hence the Lawton matrix satisfies the eigenvalue condition.

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Note: the advantage of theorem 2 is that it permits us to check test whether a Lawton matrix has a simple eigenvalue at one and all other eigenvalues of modulus less than one, not by checking all of the eigenvalues, but rather by testing a single eigenvalue σ which is known to be real, non-negative and symmetric of type (2). These known properties of σ significantly simplify the numerical task of finding this eigenvalue.

6. THE LIFTING SCHEME

We outline a single parameter form of the lifting scheme as follows:

Theorem 3: Take any initial set of real finite, balanced dual filters $\{h, \tilde{h}\}$, i.e. filters satisfying the biorthogonal constraint:

$$\tilde{H}\left(e^{i\theta}\right)\overline{H\left(e^{i\theta}\right)} + \tilde{H}\left(e^{i(\theta+\pi)}\right)\overline{H\left(e^{i(\theta+\pi)}\right)} = 1.$$

Assume that these filters generate biorthogonal Riesz bases of compactly supported wavelets. Define companion filters g and $\tilde{g}$ as follows:

$$\tilde{G}\left(e^{i\theta}\right) = e^{-i\theta}\,\overline{H\left(e^{i(\theta+\pi)}\right)}$$

$$G\left(e^{i\theta}\right) = e^{-i\theta}\,\overline{\tilde{H}\left(e^{i(\theta+\pi)}\right)}$$

then a new set of finite balanced filters $\left\{ h, \tilde{h}^{new} \right\}$, together with their companion filters $\left\{ g^{new}, \tilde{g} \right\}$, are generated as follows:

$$\tilde{H}^{new}\left(e^{i\theta}\right) = \tilde{H}\left(e^{i\theta}\right) + \tau\,\tilde{G}\left(e^{i\theta}\right)\overline{S\left(e^{i2\theta}\right)}$$

$$G^{new}\left(e^{i\theta}\right) = G\left(e^{i\theta}\right) - \tau\,H\left(e^{i\theta}\right)S\left(e^{i2\theta}\right)$$

where $S\left(e^{i\theta}\right)$ is a real trigonometric polynomial and τ is a real parameter. These new filters also satisfy the biorthogonal constraint, i.e. are dual.

The question arises as to whether, for a given real trigonometric polynomial S and real parameter τ the dual filters $\left\{ h, \tilde{h}^{new} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets. A simple necessary condition [8] is stated as follows:

Lemma 14: The dual filters $\left\{ h, \tilde{h}^{new} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets only if $S(1) = 0$.

The principle contribution of the present work (theorem 2) leads directly to the following result:

Theorem 4: Assuming $S(1) = 0$ the dual filters $\left\{ h, \tilde{h}^{new} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets for all real τ in an open interval containing 0. Moreover, this interval is characterised by the facts that it is maximal and that at any boundary points, but at no interior points, the Lawton matrix associated with $\tilde{h}^{new}$ has a symmetric eigenvalue of type (2) equal to 1.

7. AN EXAMPLE

To initialise the lifting scheme select the Haar filters $h = \left[0, \tfrac{1}{2}, \tfrac{1}{2}\right] = \tilde{h}$ and their companion filters $g = \left[0, -\tfrac{1}{2}, \tfrac{1}{2}\right] = \tilde{g}$. It is readily shown that filters h, $\tilde{h}$ satisfy the biorthogonal constraint. Note that filters h and $\tilde{h}$ are real, finite and balanced and that $H(-1) = \tilde{H}(-1) = 0$. They comprise a dual real finite pair of filters. The Lawton matrix associated with both filters is:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \tfrac{1}{2} & 0 & 0 & 0 \\ 0 & \tfrac{1}{2} & 1 & \tfrac{1}{2} & 0 \\ 0 & 0 & 0 & \tfrac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It satisfies the column sum condition and has eigenvalues $0, 0, 1, \frac{1}{2}, \frac{1}{2}$. One eigenvalue is 1. It is simple and strictly exceeds all other eigenvalues in modulus. It follows from [4] that filters $\{h, \tilde{h}\}$ generate biorthogonal Riesz bases of compactly supported wavelets. We apply the single parameter form of the lifting scheme using the fixed real trigonometric polynomial:

$$S\left(e^{i\theta}\right) = \left(-e^{i\theta} + e^{-i\theta}\right)$$

which clearly satisfies $S(1) = 0$. The new filters become:

$$\begin{aligned} h &= \left[0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right] \\ \tilde{g} &= \left[0, 0, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right] \\ \tilde{h}^{new} &= \left[0, -\tfrac{\tau}{2}, \tfrac{\tau}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{\tau}{2}, -\tfrac{\tau}{2}\right] \\ g^{new} &= \left[0, \tfrac{\tau}{2}, \tfrac{\tau}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{\tau}{2}, -\tfrac{\tau}{2}\right] \quad . \end{aligned}$$

The Lawton matrix associated with filter $\tilde{h}^{new}$, denoted Λ, is given as follows:

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 \\
-\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 \\
0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 \\
0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 \\
0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau^2}{2} & -\tau^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

This matrix is Lawton symmetric. By comparing with the canonical structure of lemma 3 we identify the sub- matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 \\ 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 \\ 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} \\ 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} \\ -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\tau^2 & 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\tau^2 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tau^2}{2} & 0 & 0 & 0 & 0 & 0 \\ -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \frac{\tau^2}{2} \\ -\tau^2 \\ -\tau+\frac{\tau^2}{2} \\ 0 \\ \frac{1}{2}+\tau-\frac{\tau^2}{2} \end{bmatrix}$$

$$c = 1 + 2\tau^2 .$$

The symmetric eigenvalues of type (2) are the eigenvalues of the reduced order matrix $\left(A + BE - 2aw^T\right) =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\tau^2 & \frac{\tau^2}{2} & 0 & 0 & 0 & 0 \\ 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} & 0 & 0 \\ 1+2\tau^2 & \frac{1}{2}+\tau-\tau^2 & 0 & -\tau+\frac{\tau^2}{2} & -\tau^2 & \frac{\tau^2}{2} \\ 2\tau^2 & \frac{1}{2}+\tau+\tau^2 & 1+4\tau^2 & \frac{1}{2}+\tau+\tau^2 & 2\tau^2 & -\tau+3\tau^2 \\ -\tau^2 & -\tau+\frac{\tau^2}{2} & 0 & \frac{1}{2}+\tau-\frac{\tau^2}{2} & 1+\tau^2 & \frac{1}{2}-\frac{\tau^2}{2} \end{bmatrix}.$$

The matrix has a symmetric eigenvalue of type (2) equal to 1 iff $\det\left(I - \left(A + BE - 2aw^T\right)\right) = 0$ where I is the identity matrix. In the present case this determinant equals the polynomial in τ:

$$\frac{1}{2}\left(1 - \frac{\tau^2}{2}\right)\left(1 + \tau^2\right)\left(1 + 2\tau - 8\tau^2\right)\left(1 + \tau\right)$$

whose roots are: $\pm\sqrt{2}, \pm i, -1, -\frac{1}{4}, \frac{1}{2}$. The maximal real open interval containing 0 with boundary points, but no interior points, in this set is given by $-\frac{1}{4} < \tau < \frac{1}{2}$. Hence, for any value of τ between $-1/4$ and $1/2$ the resulting filters $\left\{ h, \tilde{h}^{new} \right\}$ generate biorthogonal Riesz bases of compactly supported wavelets.

8. CONCLUSIONS

We have formulated a single parameter form of the lifting scheme. We have shown that the scheme generates biorthogonal filter banks having associated wavelets in $L_2(R)$ provided the parameter lies in a certain open interval and have developed a method for finding the largest such interval. We note that the parameterised lifting scheme, in conjunction with this method, yields a whole class of biorthogonal filter banks with associated wavelet bases in $L_2(R)$ and that this class is itself parameterised. Clearly one may employ a stochastic algorithm to determine the filter bank in this parameterised class which is optimal with respect to some desirable property (such as maximum energy compaction, desired shape, *etc*.).

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